Page 2:

nuclear saturation density: $\rho_0 \sim 2.8\times10^{14}\, g/cm^3$

as number density: $n_0 \sim 0.16 / fm^3$

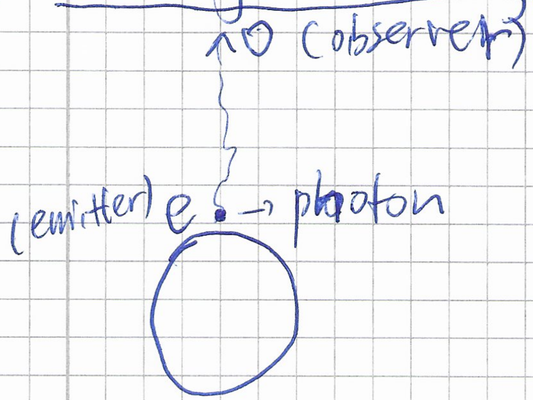
page 3

① compactness $C \equiv \dfrac{GM}{Rc^2} \sim 0.2-0.3$ for NSs

② the surface escape velocity: $v \sim \sqrt{\dfrac{GM}{R}} \sim 0.5\,c$

c ↓ speed of light

③ the surface redshift

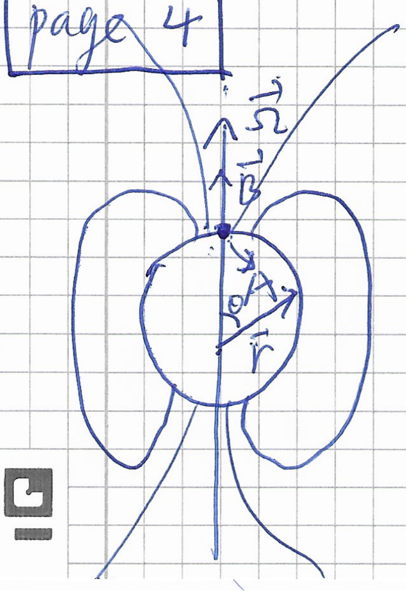


$$\frac{\omega_0}{\omega_e} = \left(\frac{g_{tt}(R)}{g_{tt}(\infty)}\right)^{\frac{1}{2}}$$

$$\Rightarrow z = \frac{\omega_e}{\omega_0} - 1 = \left(1-\frac{2GM}{Rc^2}\right)^{-\frac{1}{2}} - 1 \approx \frac{GM}{Rc^2}$$

The surface redshift is on the order of 0.2-0.3.

page 4



the charge neutrality

$$\vec{E} + (\vec{\Omega}\times\vec{r})\times\vec{B}/c = 0$$

Take the electric potential at A equals ~~zero~~ zero, then the surface electric potential:

$$\varphi = \frac{R^2\Omega B}{2c}\sin^2\theta \approx \frac{3\times10^{16} R_6^{2} B_{12}}{P}\sin^2\theta\ (V)$$

$$E \sim \varphi/R \sim 3\times10^{10}/P\ (V/cm)$$

$\boxed{\text{Page 7}}$ Low temperature physics

The Fermin temperature T_F:

$$k_B T_F \sim \frac{p_F^2}{2m}, \quad p_F = (3\pi^2 n)^{\frac{1}{3}} \hbar$$

$$\frac{p_F^2}{2m_n} = \frac{(3\pi^2 n)^{\frac{2}{3}} \hbar^2 c^2}{2 m_n c^2}$$ (consider neutron gas)

$\hbar c = 197 \text{ MeV}\cdot\text{fm}$, $\quad m_n c^2 \approx \cancel{1\text{ GeV}}\ 938 \text{ MeV}$.

$$\Rightarrow \cancel{\frac{p_F^2}{2m_n} \approx}\ k_B T_F \approx 58.4 \left(\frac{n}{n_0}\right)^{\frac{2}{3}} \text{ MeV}.$$

~~at nucl~~ $\Rightarrow T_F \approx 6.8 \times 10^{11}$ K.

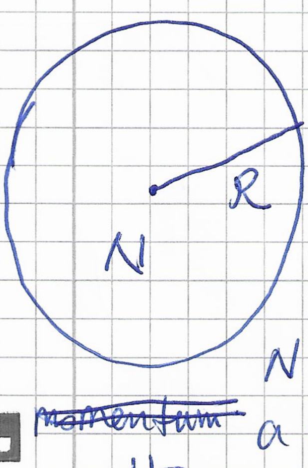
After few seconds, the star cools down below 10^{11}K

the star can be treated as zero-temperature star.

$\boxed{\text{page 15}}$ The Chandrasekhar limit.

The existence of a mass limit for a degenerate star is such an important result that we should try to understand it in a way ~~that~~ as simple as possible

the number density of Fermions $n \sim \frac{N}{R^3}$



The Fermi energy of the Fermi gas particle:

$$\begin{cases} \text{non-relativistic: } E_F \sim \hbar^2 c^2 n^{\frac{2}{3}}/m \\ \qquad \sim \hbar^2 c^2 N^{\frac{2}{3}}/mR^2 \\ \text{relativistic: } E_F \sim \hbar c n^{\frac{1}{3}} \\ \qquad \sim \hbar c N^{\frac{1}{3}}/R \end{cases}$$

N fermions in ~~momentum~~ a star of radius R.

~~the~~

The gravitational energy per fermion is:

$$E_G \sim -\frac{GMm_B}{R}$$

~~Here~~ $M = m_B N$ (note that even if the pressure comes from electrons, most of the mass is in baryons)

Equilibrium is acheived at a minimum of the total energy E

$$E = E_F + E_G \cancel{= k_1 \frac{\hbar c N^{\frac{1}{3}}}{R}}$$

$$= k_1 \hbar^2 c^2 N^{\frac{2}{3}} / mR^2 - k_2 \frac{GNm_B^2}{R} \quad \text{(non-relativistic)}$$

$$= k_1 \frac{\hbar c N^{\frac{1}{3}}}{R} - k_2 \frac{GNm_B^2}{R} \quad \text{(ultra-relativistic)}$$

Here k_1 and k_2 are constants at order of unit.

~~We start by considering the relativistic limit:~~

For ultra-relativistic case, both terms scale as $1/R$

When the sign is positive, E can be increased by increasing R, This decreases E_F and electrons tend to become nonrelativistic, with $E_F \sim P_F^2 \sim 1/R^2$

E_G dominats. E_b becomes negative, increasing to zero as $R \to \infty$, so there will be a stable equilibrium at finite value of R.

$$k_1 \frac{N^{\frac{2}{3}}}{R^2} \sim k_2 \frac{N}{R}$$

$$\Rightarrow R \sim N^{-\frac{1}{3}} \sim M^{-\frac{1}{3}} \Rightarrow R \propto M^{-\frac{1}{3}}$$

On the other hand, when the sign of E is negative, E can be decreased without bound by decreasing R — no equilibrium exists and gravitational collapse sets in.

The maximum baryon number is determined by setting $E = 0$

$$E = \frac{\hbar c N^{\frac{1}{3}}}{R} - \frac{G N m_B^2}{R} = 0$$

$$\Rightarrow N_{max} \sim \left(\frac{\hbar c}{G m_B^2}\right)^{\frac{3}{2}} \sim 2 \times 10^{57}$$

$$M_{max} \sim N_{max} \cdot m_B \sim 1.5 M_\odot$$

With the exception of numerical factors, the maximum mass of degenerate star depends only on fundamental constants.

The equilibrium radius associated with masses M approaching M_{max} is determined by the onset of relativistic degeneracy:

$E_F \gtrsim mc^2$ (m refer to the mass of particle that supplies degc. pressure

$$\frac{\hbar c N_{max}^{\frac{1}{3}}}{R} \geq mc^2$$

$$\Rightarrow R \leq \frac{\hbar c}{mc^2} \cdot \left(\frac{\hbar c}{G m_B^2}\right)^{\frac{1}{2}}$$

$$\sim \begin{cases} 5\times 10^{8}\ \text{cm}, & m = m_e \\ 3\times 10^{5}\ \text{cm}, & m = m_n. \end{cases}$$

☆ If the degenerate pressure is supplied by electrons, then the radius is around 5000 km. $\Rightarrow$ white dwarf

☆ If the degenerate pressure is supplied by neutrons then the radius is around 3km $\Rightarrow$ Neutron Stars

But why neutron stars? $\boxed{\text{page } 18}$

when the star collapses into nuclear density. the Fermi energy of electrons:

$$E_{F,e} = (3\pi^2 n)^{\frac{1}{3}} \hbar c \sim 331 \cdot \left(\frac{n}{n_0}\right)^{\frac{1}{3}} \text{MeV}.$$

The energy is much larger than the mass difference of proton and neutron, and the following process presents.

$$e^- + p \to n + \nu_e$$

electrons are killed and make neutron-rich matter

For n, p, e matter in equilibrium, we assume:

① n, p, e as non-interacting Fermi gases.

② baryons are non-relativistic while electrons are relativistic.

β-equilibrium: $\mu_n = \mu_e + \mu_p$

charge neutrality: $n_p = n_e$.

$\mu_n = m_n c^2 + E_{Fn}$, $\mu_p = m_p c^2 + E_{Fp}$

$\mu_e = E_{Fe} = \hbar c (3\pi^2 n_e)^{\frac{1}{3}}$

Homework: Further assume $n_p \ll n_n$, and $m_p = m_n = m_b$, derive the proton fraction

$$x_p = \frac{n_p}{n}$$

$$\begin{cases} \mu_n = \mu_e + \mu_p \\ n_p = n_e \end{cases} \Rightarrow \frac{p_{Fn}^2}{2m_B} = \frac{p_{Fp}^2}{2m_B} + \hbar c (3\pi^2 n_p)^{\frac{1}{3}}$$

To give structures of NSs, we need the EoS (pressure as a function of density), and the ~~gravity~~ hydroequilibrium ~~EoS~~ in GR.

For zero-temperature EoS in composition equilibrium. the first law of thermodynamics can be written as

$$d\left(\frac{\varepsilon}{n}\right) = -p\,d\left(\frac{1}{n}\right) \qquad ①$$

ε: energy density $\quad$ p: pressure.

n: number density of baryons.

$\frac{1}{n}$: volume per baryon.

according to ①: $p = -\dfrac{\partial\left(\frac{\varepsilon}{n}\right)}{\partial\left(\frac{1}{n}\right)} = n^2 \dfrac{\partial\left(\frac{\varepsilon}{n}\right)}{\partial n}$.

fluid hydrodynamics.

for perfect fluid $\quad T^{\alpha\beta} = (\epsilon + p) u^\alpha u^\beta + p g^{\alpha\beta}$.

The energy-moment conservation Eq.

$$\nabla_\alpha T^{\alpha\beta} = 0$$

(i) projection along the four-velocity:

$$u_\alpha \nabla_\beta T^{\alpha\beta} = 0 \Rightarrow \nabla_\beta (\epsilon u^\beta) = -p \nabla_\beta u^\beta$$

This is the energy conservation Eq. for the fluid element.

The projection ~~along~~ orthogonal to u^α.

$$q^\alpha{}_\gamma \nabla_\beta T^{\beta\gamma} = 0 \quad , \quad q_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta .$$

$$\Rightarrow (\epsilon + p) u^\beta \nabla_\beta u^\alpha = - q^{\alpha\beta} \nabla_\beta p .$$

This is the relativistic Euler equation

Derivation of Tolman–Oppenheimer–Volkoff EoS.

For a spherically symmetric and static solution, the line element can be written as:

$$ds^2 = -e^{\nu(r)} dt^2 + e^{\lambda(r)} dr^2 + r^2 d\Omega^2$$

If we give the spacetime metric, for the fluid element we have ~~$u^\mu u_\mu = -1$ and EoS $p = p(\epsilon)$~~ u^μ, p, ϵ (in total ~~seven~~ six variables) to solve, ~~we he~~ we have $u^\mu u_\mu = -1$, $p = p(\epsilon)$, still need four. They are:

$$\nabla_\mu T^{\mu\nu} = 0$$

The four velocity is $u^\mu = (e^{-\frac{\nu}{2}}, 0, 0, 0)$

From the relativistic Euler equation, ~~we obtain~~:

$$(\epsilon+p)\, u^{\beta}\nabla_{\beta} u^{\alpha} = -q^{\alpha\beta}\nabla_{\beta} p$$

we obtain:

$$\partial_2 p = -\frac{1}{2}(\epsilon+p)\,\partial_2 \nu$$

[Homework]

$$\Rightarrow \frac{dp}{dr} = -\frac{1}{2}(\epsilon+p)\frac{d\nu}{dr}$$

It is the relativistic version of the equations for hydrodynamical equilibrium.

still, we need find $\nu(r)$ with Einstein Eq.

$$\Rightarrow \frac{d\nu}{dr} = \frac{2(m+4\pi r^3 p)}{r(r-2m)}, \text{ where}$$

$$m(r) = 4\pi\int_0^r \varepsilon(r')\, r'^2\, dr'$$

and $$e^{\lambda(r)} = \left(1-\frac{2m}{r}\right)^{-1}$$

The quantity $m(r)$ is the "mass inside radius r"

Then

$$M = 4\pi\int_0^R \varepsilon\, r^2\, dr$$

is the total mass of the star. This includes all contributions to the total mass-energy, including the gravitational potential energy. So we usually call it $\boxed{\text{gravitational mass}}$

It's useful to compare the gravitational mass to $\boxed{\text{the baryonic mass}}$

$\Downarrow$

the mass ~~that~~ with all the particles are moved to the spatial infinity.

We have the restmass conservation equation

$$\nabla_\mu (\rho u^\mu) = 0 \quad , \quad \rho u^\mu = j^\mu$$ ~~den~~ represents the conserved baryon mass current in a local

$$\cancel{j^\mu(r) = u^\mu \rho(r) = e^{-\nu(r)/2}\rho(r)\,\delta_0^\mu}$$

inertial frame of a fluid element.

$$j^\mu = e^{-\frac{\nu(r)}{2}} \rho(r)\, \delta_0^\mu$$

The baryonic mass of the star can be derived

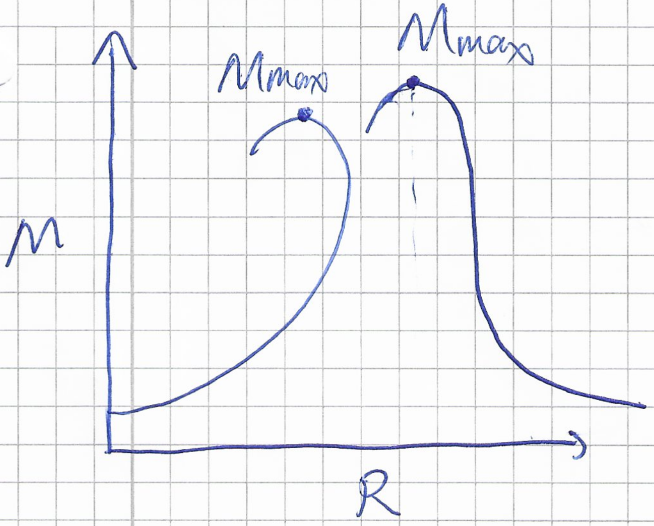
$$M_0 = \int \sqrt{-g}\, j^0\, dr\, d\theta\, d\varphi = \int_0^R 4\pi r^2 e^{\frac{\nu+\lambda}{2}} \cdot e^{-\frac{\nu}{2}} \rho\, dr$$

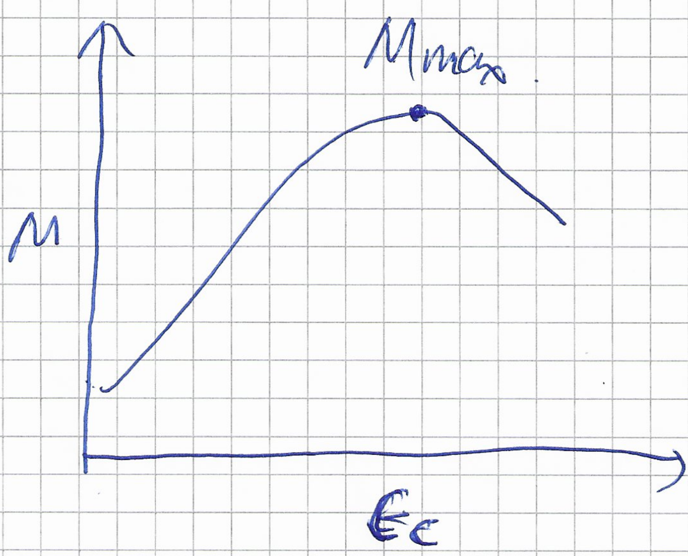
$$= \int_0^R 4\pi r^2 \frac{1}{\left(1-\frac{2m}{r}\right)^{\frac{1}{2}}} \rho(r)\, dr$$

In all, we can write.

$$\begin{cases}
\dfrac{dm}{dr} = 4\pi r^2 \varepsilon & \text{(obtain the gravitational mass)} \\[2mm]
\dfrac{dp}{dr} = -(\varepsilon+p)\dfrac{m+4\pi r^3 p}{r(r-2m)} & \text{(hydro equilibrium)} \\[2mm]
\dfrac{d\nu}{dr} = 2\cdot\dfrac{m+4\pi r^3 p}{r(r-2m)} & \Big\} \text{ determination of metric function.} \\[2mm]
e^\lambda = \dfrac{1}{1-2m/r} & \\[2mm]
\dfrac{dm_0}{dr} = 4\pi r^2 \rho \times \dfrac{1}{\left(1-\frac{2m}{r}\right)^{\frac{1}{2}}} & \Rightarrow \text{obtain the baryonic mass}
\end{cases}$$

The mass-radius curve:



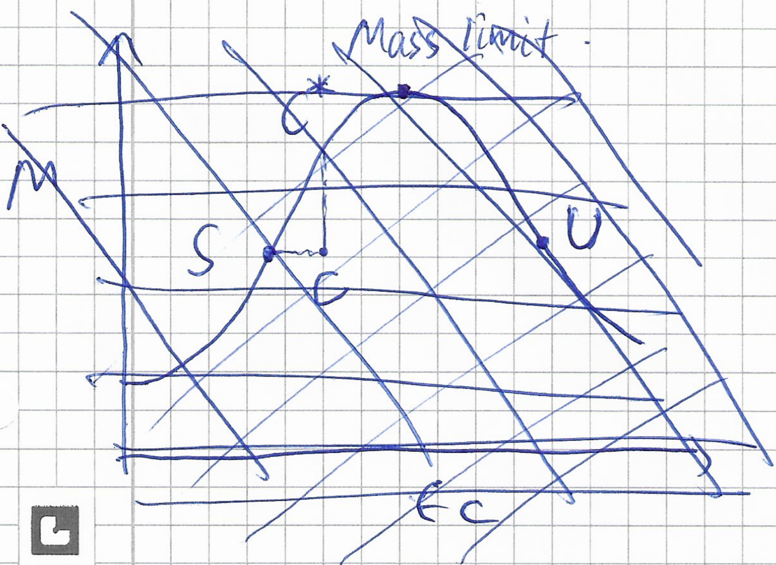


The configuration of star changes stability only at a value of the central density at which the equilibrium mass is stationary.

$$\frac{\partial M(\epsilon_c)}{\partial \epsilon_c} = 0$$

A necessary condition for stability is:

$$\frac{\partial M(\epsilon_c)}{\partial \epsilon_c} > 0$$



$$L = 4\pi R^2 \sigma T^4 , \quad F_{obs} = \frac{L}{4\pi D^2} (1+z)^{-2}$$

$$T_{obs} = T(1+z)^{-1}$$

$$F_{obs} = \frac{4\pi R^2 \sigma T_{obs}^4 (1+z)^4}{4\pi D^2} (1+z)^{-2}$$

$$= \frac{4\pi R^2 \sigma T_{obs}^4}{4\pi D^2} (1+z)^2$$

$$= \frac{4\pi R_{obs}^2 \sigma T_{obs}^4}{4\pi D^2}$$

where $R_{obs} = R(1+z) = R \cdot \left(1 - \frac{2M}{R}\right)^{-\frac{1}{2}}$